Change of Variables in a double integral. (1)

In the evaluation of double integrals (repeated), the Computational work can often be reduced by changing the Variables from one system of coordinates to another.

Procedure

Consider, $\iint_R f(x,y)\,dx\,dy$. Let the Variables be changed from x and 'y' to 'u' and 'v' by the eqnations $x=\phi(u,v)$, $y=\psi(u,v)$

Further, let the jacobian be

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

Then, it can be proved that

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\{\phi(u,v), \psi(u,v)\}\,|J|\,du\,dv \quad \text{— (1)}$$

Here, 'R' is the region in which (x,y) vary and R^* is the corresponding region in which (u,v) vary.

Double Integral in Polar form:

Using (1), we can obtain the relation connecting double integral in Cartesian form and the corresponding double integral in polar form. Let (r,θ) be the polar coordinates of the point (x,y). Then we have $x=r\cos\theta$, $y=r\sin\theta$, so that

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\left[r\cos\theta, r\sin\theta\right] r\,dr\,d\theta \quad \text{— (2)}$$

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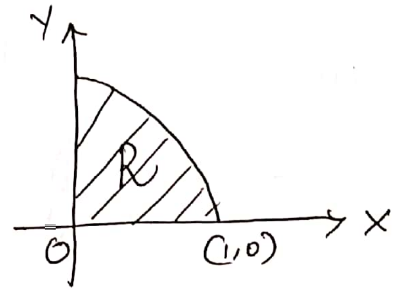
Note

1. The formula (2) is particularly useful when the region R' is bounded (in parts or whole) by a circle centered at the origin.
2. When (x, y) are changed to (r, θ); $dx\,dy$ is changed to $r\,dr\,d\theta$.

Problems

1. Evaluate $\iint xy\,dx\,dy$ over the positive quadrant bounded by the circle $x^2+y^2=1$



soln In the positive quadrant bounded by the circle $x^2+y^2=1$, the radius 'r' varies from 0 to 1 and the polar angle θ varies from 0 to $\pi/2$. On changing to polar coordinates,

$$\iint_R xy\,dx\,dy = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} (r\cos\theta)(r\sin\theta)(r\,d\theta\,dr)$$

$$= \int_0^1 r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \int_0^1 r^3 dr \int_0^{\pi/2} \frac{\sin 2\theta\,d\theta}{2}$$

$$= \frac{1}{4}\int_0^1 r^3 dr \left(-\cos 2\theta\right)_0^{\pi/2} = -\frac{1}{4}\int_0^1 r^3 \left(\cos\pi - \cos 0\right) dr$$

$$= \frac{1}{4}\int_0^1 r^3 (2)\,dr = \frac{1}{2}\int_0^1 r^3 dr = \frac{1}{2}\left[\frac{r^4}{4}\right]_0^1 = \frac{1}{8}$$

2. Evaluate $\iint xy\left(x^2+y^2\right)^{3/2} dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$ by transforming to polar coordinates.

Sol^n In the positive quadrant bounded by the circle $x^2+y^2=a^2$, the radius varies from '0' to 'a', & 'θ' varies from '0' to $\frac{\pi}{2}$. On changing to polar coordinates the integral is

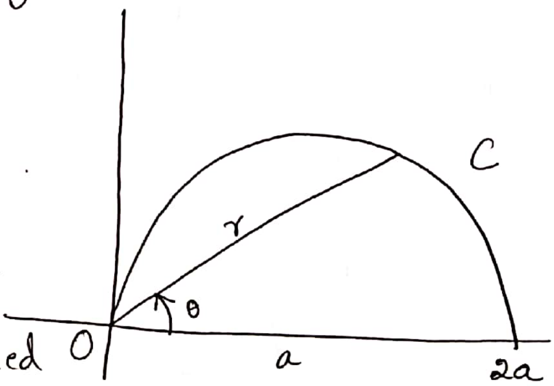
$$\iint_R xy\,(x^2+y^2)^{3/2}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta\, r\sin\theta\,(r^2)^{3/2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \int_0^a r^6\,dr = \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\left(\frac{r^7}{7}\right)_0^a = \frac{a^7}{7}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$= \frac{a^7}{7}\int_0^{\pi/2}\frac{\sin 2\theta}{2}\,d\theta = \frac{a^7}{14}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = -\frac{a^7}{28}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-a^7}{28}\left[-1-1\right] = \frac{a^7}{14}$$

(3) Evaluate $\displaystyle\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2\,dy\,dx$ by changing to polar coordinates.



sol^n Here, 'x' varies from '0' to '2a'.

'y' varies from '0' to $\sqrt{2ax-x^2}$.

(or)

y varies from '0' to $y^2=2ax-x^2$, which represents the circle 'C' centered at $(a,0)$ and passing through the origin. We observe that, 'θ' varies from '0' to $\pi/2$, 'r' varies from '0' to a point on the circle C. From the Cartesian equation of C,

We see that, $r=2a\cos\theta$

$$\therefore \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2\,dy\,dx = \int_0^{\pi/2}\int_{r=0}^{2a\cos\theta}(r^2\cos^2\theta)\,r\,dr\,d\theta$$

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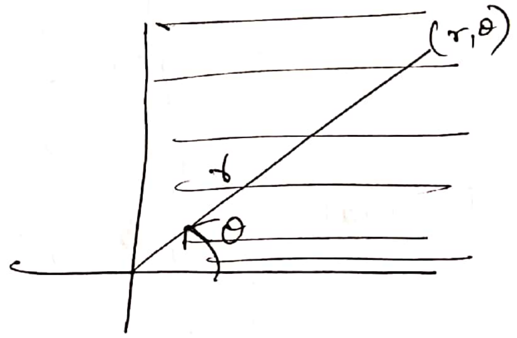
$$= \int_0^{\pi/2}\left[\cos^2\theta \int_0^{2a\cos\theta} r^3\,dr\right]d\theta = \int_0^{\pi/2}\left[\cos^2\theta \times \frac{1}{4}\left(r^4\right)\right]_0^{2a\cos\theta} d\theta$$

$$= 4a^4\int_0^{\pi/2}\cos^6\theta\,d\theta = 4a^4\,\frac{5\cdot3\cdot1}{6\cdot4\cdot2}\,\frac{\pi}{2} = \frac{5}{8}\pi a^4$$

(4) Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

Deduce that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Sol^n Both 'x' and 'y' varies from 0 to ∞. ∴ the region of integration is first quadrant. In this, quadrant 'r' varies from 0 to ∞ and 'θ' varies from 0 to $\frac{\pi}{2}$.



$$I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^\infty -\frac{1}{2}e^{-r^2}(-2r\,dr)\right]d\theta = \int_0^{\pi/2} -\frac{1}{2}\left[e^{-r^2}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \pi/4$$

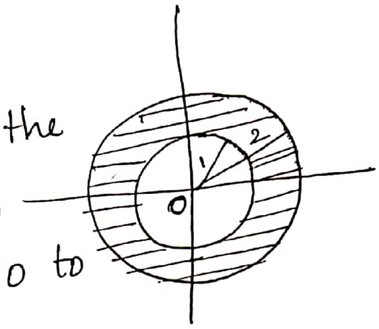
Also,

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_{x=0}^\infty\int_{y=0}^\infty e^{-x^2}e^{-y^2}\,dy\,dx = \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \left\{\int_0^\infty e^{-x^2}dx\right\}^2$$

Since, $I = \frac{\pi}{4}$ this yields $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

5. Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ where 'R' is the annular region between the circles $x^2+y^2=2$ & $x^2+y^2=1$.

Sol^n The region of integration is the annular region between the two given circles. 'θ' varies from $\theta=0$ to $\theta=2\pi$. 'r' varies from $r=1$ to $r=2$.



$$\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}\, r\,dr\,d\theta$$

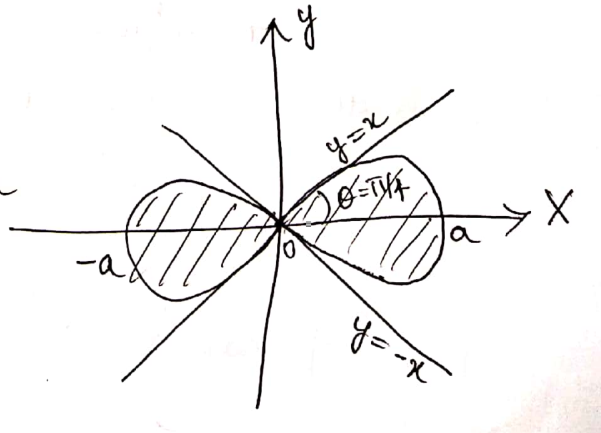
$$= \int_0^{2\pi}\left[\int_1^2 r^3\cos^2\theta\sin^2\theta\,dr\right]d\theta = \int_0^{2\pi}\cos^2\theta\sin^2\theta\left(\frac{r^4}{4}\right)_1^2 d\theta$$

$$= \frac{15}{4}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{15}{16}\int_0^{2\pi}\sin^2 2\theta\,d\theta = \frac{15}{32}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta$$

$$= \frac{15}{32}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{15}{32}(2\pi) = \frac{15\pi}{16}$$

6) Evaluate $\iint_R \frac{dx\,dy}{\sqrt{x^2+y^2+a^2}}$ where 'R' is the region bounded by the lemniscates $r^2=a^2\cos 2\theta$.

Sol^n 'R' is bounded by two loops which are symmetric about x and y axis



$$\iint_R \frac{dx\,dy}{\sqrt{x^2+y^2+a^2}} = 4\iint_{R_1} \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$$ R_1, is the part of 'R'

in the I quadrant.

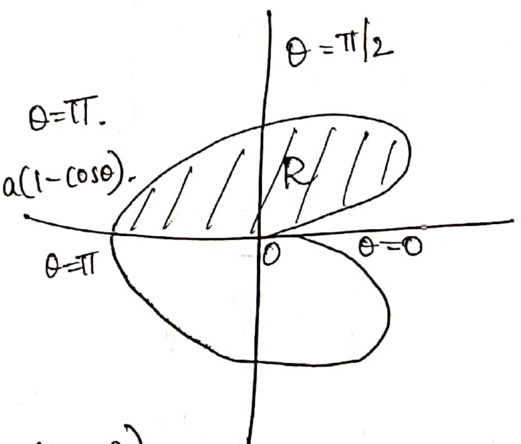
In R_1, 'θ' varies from $\theta=0$ to $\theta=\frac{\pi}{4}$ and 'r' varies from $r=0$ to $r=a\sqrt{\cos 2\theta}$.

$$\iint_R \frac{dx\,dy}{\sqrt{x^2+y^2+a^2}} = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}} = 4\int_0^{\pi/4}\left\{\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{r^2+a^2}}\right\}d\theta$$

$$= 4\int_0^{\pi/4}\left\{\sqrt{r^2+a^2}\right\}_0^{a\sqrt{\cos 2\theta}} d\theta = 4\int_0^{\pi/4}\left\{\sqrt{a^2\cos 2\theta + a^2} - a\right\}d\theta$$

$$= 4a\int_0^{\pi/4}\left(\sqrt{2}\cos\theta - 1\right)d\theta = 4a\left(1-\frac{\pi}{4}\right)$$

7. Evaluate $\iint_R r\sin\theta\,dr\,d\theta$ where 'R' is the region above the initial line bounded by the Cardiod $r=a(1-\cos\theta)$



Sol^n In R, 'θ' varies from $\theta=0$ to $\theta=\pi$.

'r' varies from $r=0$ to $r=a(1-\cos\theta)$.

$$\therefore \iint_R r\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi}\left\{\int_0^{a(1-\cos\theta)} r\,dr\right\}\sin\theta\,d\theta = \int_0^{\pi}\left\{\frac{r^2}{2}\right\}_0^{a(1-\cos\theta)} \times \sin\theta\,d\theta$$

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$$= \frac{1}{2} a^2 \int_0^{\pi} (1-\cos\theta)^2 \sin\theta d\theta$$

Put, $t = 1-\cos\theta$.

$dt = \sin\theta d\theta$

$$= \frac{1}{2} a^2 \int_0^2 t^2 dt = \frac{1}{2} a^2 \left(\frac{t^3}{3}\right)_0^2 = \frac{1}{2} a^2 \cdot \frac{8}{3} = \frac{4}{3} \cdot a^2$$

8. By changing the Variables appropriately, evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} dxdy$, where 'R' is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol^n The parametric equation of the ellipse are $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. Therefore the coordinates of a point that lies inside the ellipse boundary of the given R, may be taken as $x = au\cos t$, $y = bu\sin t$, $0 \le u \le 1$, $0 \le t \le 2\pi$.

Now,
$$\frac{\partial(x,y)}{\partial(u,t)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} a\cos t & -au\sin t \\ b\sin t & bu\cos t \end{vmatrix}$$

$$= abu(\cos^2 t + \sin^2 t) = abu$$

Therefore, on changing the Variables from (x,y) to (u,v) we get,

$$\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} dxdy = \int_{t=0}^{2\pi} \int_{u=0}^{1} \left(1 - u^2\cos^2 t - u^2 \sin^2 t\right)^{1/2} (abu) du dt$$

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$$= ab\int_0^{2\pi}\left\{\int_0^1 u(1-u^2)^{1/2}\,du\right\}dt = ab\int_0^{2\pi}dt\int_0^{\pi/2}\sin\phi\cos^2\phi\,d\phi$$

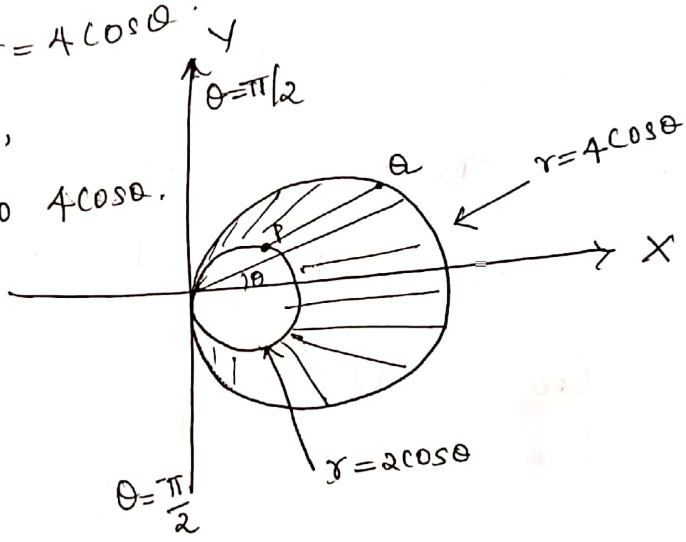
take $u=\sin\phi$

$du=\cos\phi\,d\phi$

$$= ab\int_0^{2\pi}dt\left\{-\frac{\cos^3\phi}{3}\right\}_0^{\pi/2} = ab\int_0^{2\pi}dt\left\{1/3\right\} = \frac{1}{3}ab\,(t)_0^{2\pi}$$

$$= \frac{1}{3}ab\,2\pi = \frac{2}{3}\pi ab.$$

9. Evaluate $\iint r^3\,dr\,d\theta$ over the area bounded between the circles $r=2\cos\theta$ and $r=4\cos\theta$.



Sol^n Here, shaded portion is R, and 'r' varies from $2\cos\theta$ to $4\cos\theta$. and 'θ' varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$.

$$\iint_R r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left\{\frac{r^4}{4}\right\}_{2\cos\theta}^{4\cos\theta}d\theta = \int_{-\pi/2}^{\pi/2}\frac{1}{4}\left\{256\cos^4\theta - 16\cos^4\theta\right\}d\theta$$

$$= 60\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 120\int_0^{\pi/2}\cos^4\theta\,d\theta = 120\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{45}{2}\pi$$

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Evaluate the following integrals (5) by changing them to polar coordinates

1. $$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

2) $$\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

3) $$\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$$

4) $$\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dy\, dx$$

5) $$\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

6) $$\int_{0}^{2a}\int_{0}^{\sqrt{2ax-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

7) $$\int_{0}^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy$$

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8. If R is the region in the first quadrant bounded by the Circle $x^2+y^2=2ax$ show that

$$\iint_R (x^2+y^2-a^2)\,dx\,dy = \frac{\pi a^4}{4}$$

9. If R is the region bounded by the Circle $x^2+y^2=a^2$, show that $\iint_R \sqrt{x^2+y^2}\,dx\,dy = \frac{2}{3}\pi a^3$

10. If R is the region bounded by one loop of the lemniscate $r^2=a^2\cos 2\theta$ show that

$$\iint_R \frac{dx\,dy}{x^2+y^2+a^2} = \frac{\pi}{4}\log\frac{1}{2}$$

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Applications of double integrals to find Area and Volume.

1. Computation of plane Areas:

$$\int_A f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_{y_1}^{y_2} f(x,y)\,dy\,dx = \int_c^d \int_{x_1}^{x_2} f(x,y)\,dx\,dy$$

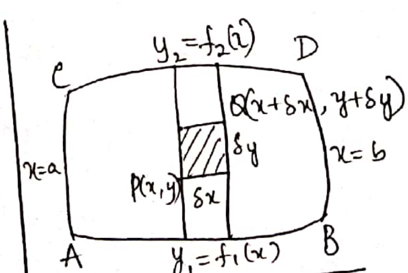
For $f(x,y)=1$, the above expression becomes

$$\int_A dA \equiv \iint_R dx\,dy = \int_a^b \int_{y_1}^{y_2} dy\,dx = \int_c^d \int_{x_1}^{x_2} dx\,dy \quad \text{—— (1)}$$

The integral $\int_A dA$ represents the total area of the plane region R over which the iterated integrals are taken.
Thus, (1) may be used to Computate the area A. in the Cartesian $dx\,dy$ is the plane area element dA in the Cartesian form.

Also, $\iint_R dx\,dy = \iint_{R^*} r\,dr\,d\theta$, $r\,dr\,d\theta$ is the plane area element in Polar form.

a) Area in Cartesian form



Let the Curves AB and CD be $y_1=f_1(x)$ and $y_2=f_2(x)$. Let the ordinates be $x=a$ and $x=b$. So the area enclosed by two curves $y_1=f_1(x)$ and $y_2=f_2(x)$ & $x=a$ and $x=b$ is ABCD. Let P & Q be two neighbouring points, then the area of $PQ=\delta x\,\delta y$.

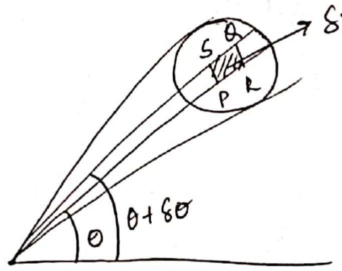
Area of the Vertical strip $= \lim_{\delta y \to 0} \sum_{y_1}^{y_2} \delta x \delta y = \delta x \int_{y_1}^{y_2} dy$

The area ABCD $= \lim_{\delta x \to 0} \sum_{a}^{b} \delta x \int_{y_1}^{y_2} dy = \int_{a}^{b} dx \int_{y_1}^{y_2} dy$

$$\boxed{\text{Area} = \int_{a}^{b} \int_{y_1}^{y_2} dx\,dy}$$

b) Area in Polar form

Consider, the area enclosed by the Curve $r = f(\theta)$. Let $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points. Draw arcs PS and QR, radii r & $r+\delta r$. $PS = r\delta\theta$, $PR = \delta r$

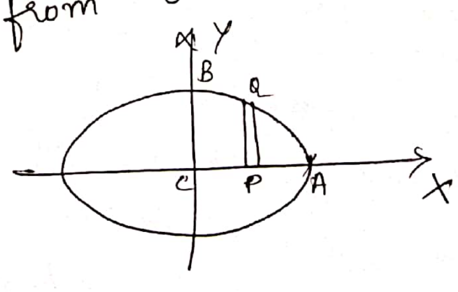


$$\boxed{\text{Area} = \iint r\,d\theta\,dr}$$

Problems

1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration

Soln We take Vertical strip, 'y' Varies from $y = 0$ to $y = \frac{b}{a}\sqrt{a^2 - x^2}$, 'x' varies from '0' to 'a'.



$$\text{Area} = 4 \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} dy\,dx$$

$$= 4 \int_0^a \left\{ \int_0^{b/a\sqrt{a^2-x^2}} dy \right\} dx = 4 \int_0^a \{y\}_0^{b/a\sqrt{a^2-x^2}} dx$$

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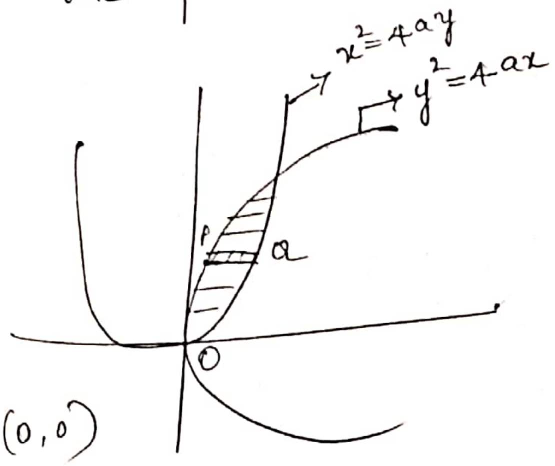
$$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = 4\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}1\right] = 4 \cdot \frac{b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \pi ab.$$

2. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.



sol^n $y^2 = 4ax$ — (1)

$x^2 = 4ay$ — (2)

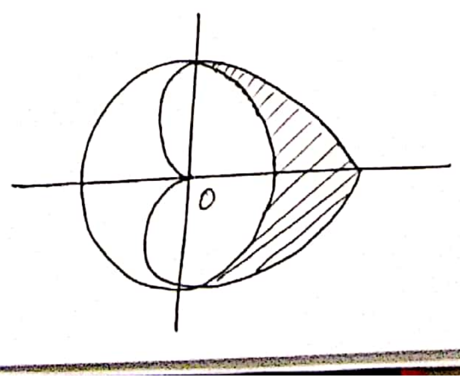
Solving (1) & (2) we get the point of intersections as $(0,0)$ & $(4a, 4a)$.

'x' varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$ and 'y' varies from 0 to $4a$.

$$\text{Area} = \int_0^{4a} dy \int_{y^2/4a}^{\sqrt{4ay}} dx = \int_0^{4a} dy\,[x]_{y^2/4a}^{\sqrt{4ay}} = \int_0^{4a}\left(\sqrt{4ay} - \frac{y^2}{4a}\right)dy$$

$$= \left(\sqrt{4a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\right)_0^{4a} = \frac{16}{3}a^2$$

3. Find the double integration the area which lies inside the Cardiod $r = a(1+\cos\theta)$ and outside the circle $r = a$



sol^n $$A = 2\int_{\theta=0}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta = a^2\left(2+\frac{\pi}{4}\right)$$

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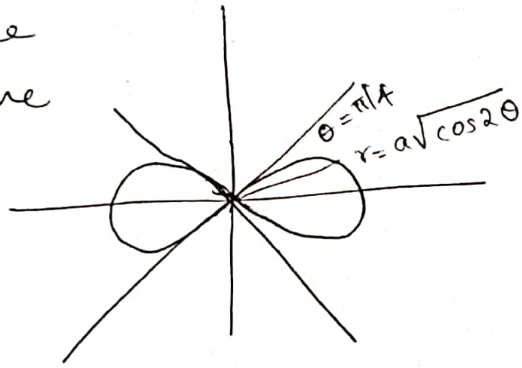
4. Find by double integration the area bounded by one loop of the lemniscate $r^2 = a^2 \cos 2\theta$

Sol^n Required Area is twice the area determined by the Curve from $\theta = 0$ to $\pi/4$



$$A = 2 \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r \, dr \, d\theta .$$

$$= a^2/2$$

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